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Painlevé analysis of the mixmaster universe

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Abstract. We analyse the behaviour of the mixmaster universe in the context of general relativity by applying methods from the geometric theory of dynamical systems in the complex plane and in particular by using what is known as the Painlevé test. We find that the singularities of the mixmaster model in the complex plane are simple poles and the resonances are $-1, 2$. This shows that chaotic behaviour may be absent in the mixmaster universe in general relativity. This analysis reinforces some recent numerical experiments which point to the same conclusion. We also Painlevé-test the Euclidean version of the Bianchi IX spacetime which includes the gravitational analogues of the Yang–Mills instanton solutions. We find that this system also has the Painlevé property and this supports previous conclusions that the Euclidean Bianchi IX field equations can be solved through Jacobi elliptic functions.

1. Introduction

The famous singularity theorems of general relativity as poineered by Hawking and Penrose in the late 1960s (see Hawking and Ellis (1973) for a review) pointed to the existence of spacetime singularities and the subsequent unpredictability of properties of spacetime near these regions of infinite density and temperature. However, the structure of spacetime near singular points naturally relates to the nature of cosmological singularities, but this issue is not addressed in the Hawking–Penrose theorems.

During the past 25 years several different techniques have been developed for examining the nature of spacetime near singularities. These include the BKL approach, Hamiltonian methods, ergodic-theoretic methods, a dynamical systems approach and numerical experiments.

The first method was developed by the Russian cosmologists Belinski, Khalatnikov and Lifshitz (hereafter BKL) (see Belinski *et al* (1970) for a review). They used certain approximation techniques based on pertubation analysis to describe the evolution of the universe towards the singularity. BKL proved that the general evolution of the universe towards the Bianchi IX singularity is oscillatory and the number of these successive series of oscillations between the three scale factors of the anisotropic Bianchi IX model tends to infinity as $t \to 0$. Misner (1969a, b, c) developed Hamiltonian methods to reformulate the problem. These methods are equivalent to the BKL analytic methods. The key feature of these Hamiltonian techniques is that the problem of solving the Einstein equations for homogeneous metrics (in our case Bianchi IX) becomes equivalent to that of tracing the motion of a particle, called the 'universe point', within a potential well, the walls of which are determined by the form of the three-curvature of the universe. These approaches revealed the extraordinary structure of the universe near the singularity and Misner coined the expression 'mixmaster universe' for this particular model of the universe. Using ergodic theory, Barrow (1982) calculated the k-entropy for the associated one-dimensional map of the mixmaster dynamical system. This analysis established a direct relation between the properties of the mixmaster universe and those of chaotic dynamical systems. Techniques from dynamical systems theory applied by Bogoyavlenski (1982) and Wainwright and Hsu (1989), to general Bianchi types and, in particular, to the mixmaster universe (Bianchi IX) gave more insight into the complicated structure of these spaces. Thus the picture emerging from all these approaches has the mixmaster universe as a purely chaotic dynamical system with its evolution as $t \to 0$ being unpredictable, stochastic and erratic.

However, difficulties in accepting such a picture may arise for the following reasons. First, recent numerical evidence (e.g. Hobill *et al* 1990) suggests that the mixmaster model has a very sensitive dependence upon initial conditions. For some initial conditions the model shows chaotic behaviour whereas other choices lead to regular solutions with no indications of chaos. This approach is based on the numerical calculation of the Liapunov exponents for the mixmaster dynamics. Most calculations suggest that the system is not chaotic all the way to the singularity. However, we point out that there is no theoretical work in published form that backs all these numerical results. Second, we note that, although the chaotic properties of the mixmaster universe were established by studying the associated one-dimensional mixmaster map, the actual system is clearly three-dimensional and for its full description one should instead consider three-dimensional Poincaré maps!

Motivated partly by the present contradictory status of the theory and partly by the suggestive results of the recent numerical experiments discussed above, we intend in this paper to look for integrability of the mixmaster dynamics through a purely analytic approach which has its root in the geometric theory of ordinary differential equations in the complex plane (see, for example, Hille 1976). Our method is based, in particular, on the so-called Painlevé test which in turn has in recent years met with a renewal of interest since it has been applied to a plethora of dynamical systems with undeniable success. For reviews of the Painlevé approach and applications, see Ramani *et al* (1989) and Bountis (1992).

We apply in this paper the Painlevé test to the mixmaster universe (Lorentzian Bianchi IX spacetime) and also to the Euclidean Bianchi IX model (see below). We find that, indeed, both systems pass the Painlevé test, i.e. the only movable singularities that both systems show in the complex plane are simple poles.

Now, it has been argued by many authors (for example, van Moerbeke 1988, Ablowitz *et al* 1980, Chang *et al* 1982 and Bountis *et al* 1982) that, if a physical system has the above-mentioned Painlevé property, then it represents a completely integrable non-chaotic dynamical system. That is, if the only movable singularities that the system can exhibit in the complex plane are simple poles (no branch points, no essential singularities), then the real-time system is completely integrable.

The organization of the paper is as follows. Section 2 serves mainly to establish notation. In section 3 we apply the Painlevé test to the Euclidean Yang–Mills equation as an example. Section 4 provides the Painlevé analysis of the mixmaster universe and we discuss our results in section 5.

2. The model and notation

The cosmological model we use is the homogeneous but anisotropic Bianchi type IX spacetime or mixmaster universe. For the classification and further properties of homogeneous but anisotropic spaces see, for example, Landau and Lifshitz (1975). For the physical motivation for studying the Bianchi IX model from the cosmological point of view, see Barrow (1982).

The metric of the homogeneous-vacuum Bianchi IX model can be taken to be (see Landau and Lifshitz 1975) of the diagonal form

$$ds^2 = -dt^2 + \gamma_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{1}$$

with

$$\gamma_{\alpha\beta} = a_1^2 l_\alpha l_\beta + a_2^2 m_\alpha m_\beta + a_3^2 n_\alpha n_\beta \tag{2}$$

where, a_1, a_2, a_3 are the scale factors along the basis vectors

$$\begin{aligned} l &= (\sin z \sin y, \cos z, 0) \\ m &= (-\cos z \sin y, \sin z, 0) \\ n &= (\cos y, 0, 1). \end{aligned} \tag{3}$$

Here, $x_i = (x, y, z)$ and the coordinate ranges are $0 \leqslant x \leqslant 4\pi$, $0 \leqslant y \leqslant \pi$ and $0 \leqslant z \leqslant 2\pi$. The three-surfaces of the homogeneous type IX model are closed and have finite volume given by $V = 16\pi^2\alpha_1\alpha_2\alpha_3$.

Then the Einstein equations for the diagonal type IX model can be written as follows (Landau and Lifshitz 1975, Barrow 1982):

$$\begin{aligned} 2\alpha_1'' &= (a_2^2 - a_3^2)^2 - a_1^4 \\ 2\alpha_2'' &= (a_1^2 - a_3^2)^2 - a_2^4 \\ 2\alpha_3'' &= (a_1^2 - a_2^2)^2 - a_3^4 \end{aligned} \tag{4}$$

$$\tfrac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3)'' = \alpha_1'\alpha_2' + \alpha_1'\alpha_3' + \alpha_2'\alpha_3' \tag{5}$$

where $' := d/dt$. Here we adopt the so-called logarithmic time which has been used extensively by many authors over the years and is related to the synchronous time coordinate t by the equation

$$d\tau = \frac{dt}{a_1 a_2 a_3}. \tag{6}$$

Typically the volume $a_1a_2a_3$ grows like t and so (6) gives $\tau \sim \ln t$. The scale factors α_1, α_2, α_3 are the logarithms of a_1, a_2, a_3, i.e, $\alpha_1 = \ln a_1 \Longleftrightarrow a_1 = e^{\alpha_1}$, etc. In τ-time the cosmological singularity is at $\tau = -\infty$. We note also that the field equations (4) are the equations $R_1^1 = 0$, $R_2^2 = 0$, $R_3^3 = 0$ and equation (5) is the equation $-R_0^0 = 0$.

For future reference we give the Euclidean version of the field equations (4). They are

$$\begin{aligned} (\log a_1^2)_{zz} &= a_1^4 - (a_2^2 - a_3^2)^2 \\ (\log a_2^2)_{zz} &= a_2^4 - (a_1^2 - a_3^2)^2 \\ (\log a_3^2)_{zz} &= a_3^4 - (a_1^2 - a_2^2)^2. \end{aligned} \tag{7}$$

This system, which is the analytic continuation of the field equations (4) to complex time, was studied in Belinski *et al* (1978) and Berger and Spero (1983) in the hope of finding the gravitational analogue of the instanton solutions of the classical Yang–Mills theory. Equations (7) have the form of the Euler equations for an asymmetric top but with unusual moments of inertia (Belinski *et al* 1978). The general solution of this system can be found in terms of Jacobi elliptic functions. It is interesting to note (Cotsakis 1990) that these equations have exactly the same form as those investigated by Kovalevskaya (1889).

3. Painlevé method and the Yang–Mills field

Many known integrable, non-chaotic dynamical systems have the Painlevé property. For example, the integrable cases of the Hénon–Heiles system are all known to be of Painlevé type (Chang *et al* 1982, Tabor 1989) and also the integrable cases of the Toda lattice with $N = 2$ and 3 can all be recovered by using the Painlevé test (Bountis *et al* 1982).

As an example to see how the Painlevé method works consider the system (Cotsakis 1990)

$$x'' = xy^2 \qquad y'' = x^2y. \tag{8}$$

This system, under the assumption that the potentials depend only on time, and under a more restrictive ansatz (Matinyan *et al* 1981, Savvidy 1984), corresponds to the Euclidean Yang–Mills equations. It is also known to be integrable since it contains the famous instanton solutions first discovered by Polyakov (1975) and Belavin *et al* (1975). We are interested to see whether or not the system (8) satisfies the Painlevé property. For this, let us first find the dominant behaviour. We substitute into (8) the forms

$$x = \alpha z^p \qquad y = \beta z^q \tag{9}$$

and obtain

$$p = q = -1 \tag{10}$$

$$\alpha^2 = \beta^2 = 2. \tag{11}$$

To determine the resonances we substitute

$$x = \alpha z^{-1} + \gamma z^{r-1} \qquad y = \beta z^{-1} + \delta z^{r-1} \tag{12}$$

into (8) and equate the terms linear in γ and δ (all terms are dominant) to zero to obtain

$$\begin{bmatrix} (r-1)(r-2) - \beta^2 & -2\alpha\beta \\ -2\alpha\beta & (r-1)(r-2) - \alpha^2 \end{bmatrix} \begin{bmatrix} \gamma \\ \delta \end{bmatrix} = 0. \tag{13}$$

Equation (13) has a non-trivial solution if

$$r = -1, 4, \tfrac{1}{2}(3 \pm i\sqrt{5}). \tag{14}$$

The presence of the complex resonances indicates that the system (8) does not possess the Painlevé property. However, the resonance at $r = 4$ does lead to the two-parameter solution

$$\begin{aligned} x &= \frac{\alpha}{z}\left\{1 + az^4 + \frac{1}{6}(az^4)^2 + \frac{1}{26}(az^4)^3 + \ldots\right\} \\ y &= \frac{\beta}{z}\left\{1 + az^4 + \frac{1}{6}(az^4)^2 + \frac{1}{26}(az^4)^3 + \ldots\right\} \end{aligned} \tag{15}$$

where a is an arbitrary constant, i.e. up to a possible sign difference $x = y$ ((8) is invariant under the discrete transformation $x \to -x$, $y \to -y$). In this case the solution of (8) can be expressed in terms of Jacobi elliptic functions.

We now turn to the main part of the paper which is the Painlevé analysis of the Bianchi IX spacetime in both the Euclidean and the Lorentzian versions.

4. Painlevé analysis of the mixmaster universe

We first focus on the Euclidean Bianchi IX model. As we shall shortly see, the Painlevé test for the mixmaster universe can be applied as in the Euclidean case.

Under the change of variables

$$a_1^2 = u \qquad a_2^2 = v \qquad a_3^2 = w \tag{16}$$

the Euclidean Bianchi IX equations become

$$uu'' - u'^2 = u^4 - u^2(v-w)^2 \qquad \text{et cyc.} \tag{17}$$

Substituting into (15) the forms

$$u = \alpha z^p \qquad v = \beta z^q \qquad u = \gamma z^r \tag{18}$$

we obtain

$$\alpha^2 p(p-1)z^{2p-2} - \alpha^2 p^2 z^{2p-2} = \alpha^4 z^{4p} - \alpha^2 z^{2p}(\beta z^q - \gamma z^r)^2 \qquad \text{et cyc.} \tag{19}$$

Balancing these terms to leading order we find that

$$2p - 2 = 4p = 2p + 2q = 2p + q + r = 2p + 2r \qquad \text{et cyc.} \tag{20}$$

From these equations we easily see that the only choice is

$$p = q = r = -1. \tag{21}$$

The calculation of the coefficients α, β, γ in (18) is somewhat more complicated. From (19) we have

$$\alpha^2 = \alpha^4 - \alpha^2(\beta - \gamma)^2 \qquad \text{et cyc.} \tag{22}$$

After some manipulation we find that the only possible choice is

$$\alpha = \beta = \gamma = \pm 1. \tag{23}$$

Therefore the result for the dominant behaviour to leading order in powers of z is

$$u = \pm z^{-1} \qquad v = \pm z^{-1} \qquad w = \pm z^{-1}. \tag{24}$$

Now we proceed to find the resonances. For this we substitute

$$u = \alpha z^{-1} + a z^{r-1} \qquad v = \alpha z^{-1} + b z^{r-1} \qquad w = \alpha z^{-1} + c z^{r-1} \tag{25}$$

where $\alpha^2 = 1$, into the field equations (17). We find that $r = -1, 2$. The coefficients a, b and c are arbitrary and we find the following four-parameter solutions

$$\begin{aligned}
u &= \frac{\alpha}{z}\left\{1 + az^2 + \frac{1}{10}[7a^2 - (b-c)^2]z^4 \right.\\
&\qquad \left. + \frac{1}{70}[31a^3 - (4(b+c) + 7a)(b-c)^2]z^6 + \ldots\right\}\\
v &= \frac{\alpha}{z}\left\{1 + bz^2 + \frac{1}{10}[7b^2 - (c-a)^2]z^4 \right.\\
&\qquad \left. + \frac{1}{70}[31b^3 - (4(c+a) + 7b)(c-a)^2]z^6 + \ldots\right\}\\
w &= \frac{\alpha}{z}\left\{1 + cz^2 + \frac{1}{10}[7c^2 - (a-b)^2]z^4 \right.\\
&\qquad \left. + \frac{1}{70}[31c^3 - (4(a+b) + 7c)(a-b)^2]z^6 + \ldots\right\}.
\end{aligned} \tag{26}$$

Thus the Euclidean Bianchi IX model has the partial Painlevé property. Now we turn to the mixmaster universe which corresponds to the analytic continuation of the field equations for the Euclidean case. The analysis of this case proceeds in direct analogy to the Euclidean case discussed above. Under the transformation (14) the field equations (4) become

$$uu'' - u'^2 = -u^4 + u^2(v-w)^2 \qquad \text{et cyc.} \tag{27}$$

The Painlevé analysis proceeds as above and we simply quote the results:

$$\begin{aligned} u &= \frac{\alpha}{z}\left\{1 + az^2 - \frac{1}{18}[a^2-(b-c)^2]z^4 \right.\\ &\qquad \left. - \frac{1}{162}[3a^3-(7a-3(b+c))(b-c)^2]z^6 + \dots\right\} \\ v &= \frac{\alpha}{z}\left\{1 + bz^2 - \frac{1}{18}[b^2-(c-a)^2]z^4 \right.\\ &\qquad \left. - \frac{1}{162}[3b^3-(7b-3(c+a))(c-a)^2]z^6 + \dots\right\} \\ w &= \frac{\alpha}{z}\left\{1 + cz^2 - \frac{1}{18}[c^2-(a-b)^2]z^4 \right.\\ &\qquad \left. - \frac{1}{162}[3c^3-(7c-3(a+b))(a-b)^2]z^6 + \dots\right\}. \end{aligned} \tag{28}$$

Equation (28) is a four-parameter solution of system (27) and means that the mixmaster universe passes the Painlevé test and possesses the partial Painlevé property in the same way as the Bianchi IX model does.

5. Discussion

The results of the previous section show clearly that the two versions of the Bianchi IX spacetime, Euclidean and Lorentzian, possess the partial Painlevé property. There is a vast literature on the connection between the Painlevé property of the solutions of a given dynamical system and integrability (see Bountis (1992) for a recent review and references). However, it is well known that the Painlevé test is, in general, only a necessary condition for the integrability of a physical system (see also in this connection Cotsakis (1990)).

With this reservation in mind we may conclude that our results imply that the mixmaster universe may not be completely chaotic for, if it were, it should not have passed the Painlevé test. We notice that all previous claims relating to the chaoticity in the mixmaster dynamics are either approximate (BKL analytic solutions) or do not take into account the full number of degrees of freedom. For instance, the well known ergodic results (Barrow 1982) that gave the first connection between the mixmaster oscillations and the occurrence of chaos in the mixmaster universe were based on the assumption that the Poincaré map for the mixmaster dynamics was one-dimensional. However, the full system has clearly three degrees of freedom and so it cannot be fully described by a one-dimensional return mapping.

Almost all recent numerical experiments point to the fact that the Liapunov exponents for the mixmaster universe go quickly to zero as we approach the initial singularity at $t = 0$.

This means that the chaotic properties of the mixmaster universe disappear as $t \to 0$ and a monotonic non-chaotic approach to the singularity occurs.

On the other hand, our analysis is not based on any approximations like those used in constructing analytic solutions, maps etc and, in this sense, it complements and reinforces the numerical results and points to the integrability of the mixmaster dynamics. If the mixmaster universe has the partial Painlevé property, this may imply that the full system is not chaotic near the initial singularity. Indeed our results support the numerical evidence of Hobill *et al* (1990) mentioned in the introduction. The partial Painlevé property suggests that the mixmaster universe is integrable on a hyperline in the six-dimensional phase space.

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